

Example 1

Prove that $f(n) = 6n^2 + 2n + 15$ is odd for all $n \in \mathbb{Z}^+$.

Proof by induction:

Predicate. Define $P(n)$ as the predicate that $f(n)$ is odd.

Base Case. We prove $P(1)$.

$$2(1) + 15 + 6(1)^2 = 2 + 15 + 6 = 23. \text{ Since } 23 \text{ is odd, } P(1) \text{ is true.}$$

Inductive Hypothesis. Assume $P(k)$ for some $k \in \mathbb{Z}^+$.

That means that $6k^2 + 2k + 15 = 2m + 1$ for some $m \in \mathbb{Z}$.

Inductive Step. We will now prove $P(k + 1)$.

$$\begin{aligned} 6(n + 1)^2 + 2(n + 1) + 15 &= (6n^2 + 12n + 6) + (2n + 2) + 15 \\ &= (2n + 15 + 6n^2) + 12n + 8 \\ &= (2m + 1) + 12n + 8 \\ &= 2(m + 6n + 4) + 1 = 2j + 1 \end{aligned}$$

Since $j = m + 6n + 4 \in \mathbb{Z}$, $f(k + 1)$ is odd. Thus, $P(k) \Rightarrow P(k + 1)$.

Conclusion. Since $P(1)$ and $P(k) \Rightarrow P(k + 1)$, $P(k)$ for all $k \in \mathbb{Z}^+$.

Example 2

Prove that if $x \geq -1$, then $(1 + x)^n \geq 1 + nx$ for all integers $n \geq 1$.

Proof by induction:

Predicate. Let $P(n)$ be that statement that $(1 + x)^n \geq 1 + nx$ for all $x \geq -1$.

Base Case. First, we prove $P(1)$.

$$(1 + x)^1 \geq 1 + x \text{ for all } x \geq -1. \text{ Therefore } P(1) \text{ holds.}$$

Inductive Hypothesis. Assume that $P(k)$ is true for some natural number $k \geq 1$.

Inductive Step. We will prove $P(k + 1)$.

From the IH, we know that for all $x \geq -1$, $(1 + x)^k \geq 1 + kx$.

Multiplying both sides by $1 + x \geq 0$ gives:

$$\begin{aligned} (1 + x)^{k+1} &\geq (1 + kx)(1 + x) \\ &= 1 + (k + 1)x + kx^2 \\ &\geq 1 + (k + 1)x \end{aligned}$$

So $P(k) \Rightarrow P(k + 1)$, and the inductive step is proved.

Conclusion. Because $P(1)$ is true and $P(k)$ implies $P(k + 1)$ for all $k \geq 1$, $P(n)$ is true for all $n \geq 1$.

Example 3

The substrings of a string $a_1 a_2 \dots a_n$ are all strings of the form $a_i a_{i+1} \dots a_j$ for all i, j such that $0 \leq i \leq j \leq n$ and where a_0 is the empty string. For example, “”, “ba”, and “banana” are substrings of the string “banana,” while “bann” is not. “b” can be represented as either $a_0 a_1$ or a_1 , but these different representations are not distinct substrings. “an” can be represented as either $a_1 a_2$ or $a_3 a_4$; these are distinct substrings.

More Induction

Prove that the number of distinct substrings of a string $a_1a_2 \dots a_n$ is

$$\frac{n(n+1)}{2} + 1.$$

Proof by induction:

Predicate. Let $P(k)$ be the predicate “all strings $a_1a_2 \dots a_k$ have exactly $\frac{n(n+1)}{2} + 1$ distinct substrings.”

Base Case. We show $P(0)$.

A string of length 0 is simply the empty string a_0 . The only substring of the empty string is the empty string, meaning that there is only one substring. $\frac{0*1}{2} + 1 = 1$. Thus, $P(0)$ is true.

Inductive Hypothesis. Assume that $P(k)$ is true for some $k \geq 0$; that is, for all strings $a_1a_2 \dots a_k$ there are exactly $\frac{k(k+1)}{2} + 1$ distinct substrings.

Inductive Step. Now $P(k+1)$ must be shown to hold, meaning that for all strings $a_1a_2 \dots a_k a_{k+1}$ there are exactly $\frac{(k+1)(k+2)}{2} + 1$ substrings.

The string $a_1a_2 \dots a_k a_{k+1}$ has exactly one more character than some string $a_1a_2 \dots a_k$. All substrings of the shorter string are substrings of the longer, and for the longer there are also $k + 1$ distinct new substrings: all substrings of the shorter string that end in a_k with a_{k+1} added to the end, as well as just a_{k+1} .

By inductive hypothesis, there are $\frac{k(k+1)}{2} + 1$ substrings of the shorter string; thus, there are $\frac{k(k+1)}{2} + 1 + k + 1$ substrings of the longer.

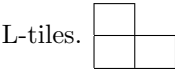
$$\begin{aligned} \frac{k(k+1)}{2} + 1 + k + 1 &= \frac{k(k+1)}{2} + 1 + \frac{2(k+1)}{2} \\ &= \frac{k(k+1) + 2(k+1)}{2} + 1 \\ &= \frac{k^2 + 3k + 2}{2} + 1 \\ &= \frac{(k+1)(k+2)}{2} + 1 \end{aligned}$$

This is what was expected and required; clearly the number of substrings of all strings $a_1a_2 \dots a_k a_{k+1}$ is $\frac{(k+1)(k+2)}{2} + 1$, so $P(k)$ implies $P(k+1)$.

Conclusion. Thus, because $P(0)$ is true and $P(k)$ implies $P(k+1)$ for all $k \geq 0$, $P(n)$ is true for all $n \geq 0$.

Example 4

Prove that for $n \in \mathbb{Z}^+$, a $2^n \times 2^n$ chessboard with any one square removed can be tiled by these 3-square L-tiles.



Proof by induction:

Predicate. Let $P(n)$ be the predicate that a $2^n \times 2^n$ chessboard with any one square removed can be tiled by the 3-square L-tiles.

Base Case. We prove $P(1)$.

For $n = 1$, we have a 2×2 board with 1 square removed, which can be tiled by 1 L-tile.

Inductive Hypothesis. Assume $P(k)$ holds for some $k \geq 1$.

More Induction

Inductive Step. We prove $P(k + 1)$.

Divide the $2^{k+1} \times 2^{k+1}$ board as follows, where A,B,C,D are each a $2^k \times 2^k$ board.

A	B
C	D

Without loss of generality, suppose that the one square has been removed from B.

Then by the inductive hypothesis, B can be tiled.

Remove the center corners of A,C, and D, such that each of their remainders can be tiled by the inductive hypothesis. Tile the remaining 3 squares in the center with a single L-tile, and we have completed the tiling.

Conclusion. Since the base case $P(1)$ holds and we have shown that $P(k) \Rightarrow P(k + 1)$, $P(n)$ holds for $n \geq 1$. □