Induction

Requirements

1. Formally **define the predicate** that will be proved inductively.
2. Prove that the predicate holds in the **base case**.
3. Formally state the **inductive hypothesis**.
4. Assume the inductive hypothesis, and prove the **inductive step**.
5. **Conclude** that the predicate holds in general.

Example

Prove that for all integers $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof

Predicate. Let $P(n)$ be the predicate "$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$."

Base case. Because $1 = \frac{1(2)}{2}$, we see that the base case $P(1)$ holds.

Inductive Hypothesis. Assume the inductive hypothesis, that for a particular k, $P(k)$ is true. That is, assume

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Inductive Step. We must prove $P(k+1)$. That is, we must prove that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

Invoking the inductive hypothesis, we can compute the sum as follows.

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+2)(k+1)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

This is exactly what we needed to show.

Conclusion. Because $P(1)$ is true, and because, for all k, $P(k)$ implies $P(k+1)$, we conclude that $P(n)$ is true for all integers $n \geq 1$. $\square$