Proof by Contradiction

Requirements

1. State the assumption that will lead to a contradiction.
2. Show that the assumption leads to a contradiction.
3. Conclude that the assumption is false.

Example

Prove that there are infinitely many primes.

Proof. Assume, for the sake of contradiction, that there are finitely many primes. Let $k$ be the number of primes, and let the primes be $p_1, p_2, \ldots, p_k$. Note that 1 is not prime, so each prime is greater than 1. Consider the number

$$N = p_1 p_2 \ldots p_k + 1$$

$N$ is either prime or composite.

Case 1: $N$ is prime.

For each $p_i$, $N > p_i$. Thus, there so no $p_i$ such that $N = p_i$. So $N$ is not one of the $k$ primes, so there are at least $k + 1$ primes. This is a contradiction, because we assumed there were only $k$ primes.

Case 2: $N$ is composite.

$N$ must have some prime factor. But note that for each $p_i$, $N = mp_i + 1$ for some integer $m$. Since $p_i > 1$, $p_i \nmid N$. So whatever prime factor $N$ has, it cannot be one of the $k$ primes on the list. This is a contradiction, because we assumed there were only $k$ primes.

Each case leads to a contradiction, so our original assumption that there were finitely many primes was wrong, and there must be infinitely many primes.