Division into Cases

Requirements

1. Show that the proposition always falls into one of a few cases.
2. List the cases.
3. Under each case, give a proof that the proposition holds for that case.
4. Conclude that the overall proposition holds.

Example

Prove that the square of any odd integer has the form $8m + 1$ for some integer m.

Proof. Suppose n is an odd integer. By the Quotient-Remainder Theorem, n can be written as $4q+r$, where q and r are integers and $0 \leq r < 4$. Because $4q$ and $4q+2$ are even, n must be of the form $4q+1$ or $4q+3$.

Case 1: $n = 4q + 1$.

Proof of Case 1:

$$\begin{aligned} n^2 &= (4q+1)^2 \\ &= 16q^2 + 8q + 1 \\ &= 8(2q^2 + q) + 1 \end{aligned}$$

Let $m = 2q^2 + q$. m is an integer, because 2 and q are integers and the sums and products of integers are integers. Substituting, we get $n^2 = 8m + 1$ where m is an integer.

Case 2: $n = 4q + 3$.

Proof of Case 2:

$$\begin{aligned} n^2 &= (4q+3)^2 \\ &= 16q^2 + 24q + 9 \\ &= 8(2q^2 + 4q + 1) + 1 \end{aligned}$$

Let $m = 2q^2 + 4q + 1$. m is an integer, because it is the sum of products of integers. Substituting, we get $n^2 = 8m + 1$ where m is an integer.

Cases 1 and 2 show that for any odd integer n, $n^2 = 8m + 1$ where m is an integer. □