Requirements

1. Formally define the two sets claimed to have equal cardinality.
2. Formally define a function from one set to the other.
3. Prove that the function is bijective by proving that it is both injective and surjective.
4. Conclude that since a bijection between the 2 sets exists, their cardinalities are equal.

Example

Prove that the number of bit strings of length n is the same as the number of subsets of the set of integers $\{1, 2, ..., n\}$.

Proof. The set of bit strings of length n can be expressed as $\{0, 1\}^n$. In this representation, each string becomes an n-tuple, generically $(a_1, a_2, ..., a_n)$

The set of subsets of the set $A = \{1, 2, ..., n\}$ is just the power set of A, or $\mathcal{P}(A)$.

In order to prove that the number of bit strings of length n is equal to the number of subsets of A, we will construct a bijection between these 2 sets.

Define the function $f : \{0, 1\}^n \to \mathcal{P}(A)$ where

$$f((a_1, a_2, ...a_n)) = \{i | a_i = 1\}$$

This function maps the string to the set of all positions of 1's in the string. For example, for $n = 5$, the bit string 10011, represented by (1,0,0,1,1), would map to $\{1, 4, 5\}$.

We now show that f is both injective and surjective, and therefore bijective.

Injective:

Suppose that two bit strings x and y both map to a single subset $S \in \mathcal{P}(A)$. Since S is defined by the positions of 1's in the bit string, if $f(x) = f(y) = S$, x and y must have 1's in exactly the same positions. But since bit strings only contain 0's and 1's, and x and y are the same length n, x and y must be identical. Therefore, since at most 1 bit string can map to a single subset, f is injective.

Surjective:

Consider a subset $S \in \mathcal{P}(A)$. S contains only integers between 1 and n, inclusive. Then we can construct an n-tuple $x = (x_1, x_2, ..., x_n)$ such that all for all $i \in S$, $x_i = 1$, and all others are 0. By the definition of our function, we have constructed exactly the n-tuple x that will map to S. Thus, since all subsets S are mapped to by some n-tuple x representing a bit string of length n, f is surjective.

Since f is both injective and surjective, f is bijective. We have therefore created a bijection between the set of bit strings of length n and the subsets of the set of integers $\{1, 2, ..., n\}$, so these 2 sets have equal cardinality. □