Bidirectional Proof

Requirements

1. Write two separate proofs, one for each direction.
2. Clearly state which direction you are proving.
3. Conclude that having proved both directions, the statement holds.

Example

Let n and m be integers. Prove that $|m| = |n|$ *if and only if* $m \mid n$ and $n \mid m$.

Proof. We prove the statement in both directions.

Forward direction: If $|m| = |n|$, then $m|n$ and $n|m$.

If $|m| = |n|$, then either $m = n$ or $m = -n$.

- **Case 1:** $m = n$.
  Then $n = 1m$, where 1 is an integer, so $m|n$. Similarly, $m = 1n$, where 1 is an integer, so $n|m$.
- **Case 2:** $m = -n$.
  Then $m = (-1)n$, where -1 is an integer, so $n|m$. It also holds that $n = -m$, so $n = (-1)m$, where -1 is an integer, so $m|n$.

Since in both cases, $m|n$ and $n|m$, it is true that $m|n$ and $n|m$ when $|m| = |n|$.

Backward direction: If $m|n$ and $n|m$, then $|m| = |n|$.

Since $m|n$, there exists an integer c such that $n = cm$. Since $n|m$, there exists an integer k such that $m = kn$. Plugging in, we get that $n = cm = c(kn)$, so $ck = 1$. Since $ck = 1$, where k is an integer, $c|1$. But the only divisors of 1 are ± 1, so $c = \pm 1$. Plugging in again, we get that $n = cm = (\pm 1)m = \pm m$, so $|n| = |m|$.

Having proved both directions, we conclude that the statement is true. □